CREATION OF OVERLAY TILINGS THROUGH DUALIZATION OF REGULAR NETWORKS

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A paper by Stampfli discusses creating quasiperiodic tilings from periodic tesselations, "if the tessellation can be reproduced from an appropriate dual lattice," [2] but contains no results concerning exactly when such tessellations can be used to create tilings. We more fully develop this idea into a more general method of creating a variety of attractive tilings using dualization.

First steps

Tilings of the plane have long been appreciated as an art form, and have been used in a wide variety of decorative applications, from fabrics to architecture. In this paper, we develop a method of creating tilings based on an underlying network (a mathematical structure similar to a graph) through a dualization technique. We can show how a common class of tilings can be constructed through this method, and how the networks used for doing that can be combined in a straightforward manner to produce interesting new varieties of tiling.

We begin with definition of a network. A *network* (see Figure 1) is a collection of points in the plane, called *vertices*, linked by straight-line segments called *edges.* Edges do not intersect except at vertices, where the endpoints of two edges may be coincident. A network is thus like a planar graph that has been embedded in the plane. Since each vertex has a specific location, it makes sense to talk about quantities like the length of an edge, or the angle between two edges. A network divides the plane up into a number of bounded regions called *faces.* (Our definition of a network has also been called a *planar subdivision* in computational geometry literature.)

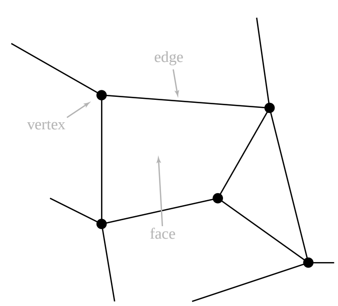


Figure 1. A portion of a sample network.

Our dualization method is going to associate a tile with each vertex of a network. Vertices that are adjacent (connected by an edge) in the network will correspond to pairs of tiles that share an edge in the tiling. Each face of the network will correspond to a tiling vertex, where all the tiles that are duals of vertices on the face come together at a single point.

One way of accomplishing this would be to pick a point within each face of the network (such as the centroid of the face), and then connect up points that correspond to neighboring faces. This means that the shape of the tile corresponding to a given vertex depends not only on that vertex but also on the characteristics of the surrounding neighborhood. Since the number of different neighborhoods is large, the resulting tiling will have a wide variety of shapes as tiles. This is somewhat unsatisfactory. We seek a solution that relies less on the geometry of the network and more on its topology, and provides better control over the shapes of the tiles that form the tiling.

Defining the dual of a vertex

Our solution comes from noticing that the total of the angles between edges surrounding any given vertex must total 360 degrees. The sum of the exterior angles of any polygon must also total 360 degrees. (See Figure 2.) This suggests constructing a vertex's tile by using the angles around the vertex as the exterior angles of a polygonal tile. Tiles constructed in this way can always be oriented so that each edge of the tile is perpendicular to its corresponding network edge. (Clearly we can orient the tile so that this is true for one edge, and since the angles between successive network edges round a vertex are equal to the angles between successive tile edges, the perpendicularity is maintained all the way around the tile.) We call this the *standard orientation* of the dual tile (see Figure 3). Once the created tiles are oriented properly, it can be seen that the tiles corresponding to vertices surrounding a face can be made to fit together at a single point, which will be the tiling vertex corresponding to that face. An example of this is shown in Figure 4.

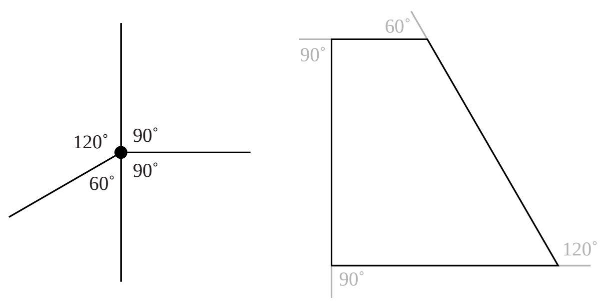


Figure 2. Two quantities that must always add up to 360 degrees: on the left, the sum of angles surrounding a vertex. On the right, the sum of exterior angles of a polygon. This suggests a method of creating polygonal tiles corresponding to the vertices of a network.

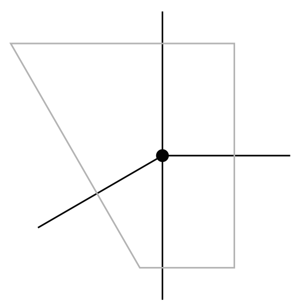


Figure 3. A vertex, with a dual polygon shown in standard orientation. Note that each edge of the polygon is perpendicular to an outgoing edge, and that the sequence of inter-edge angles around the vertex is repeated in the polygon's exterior angles.

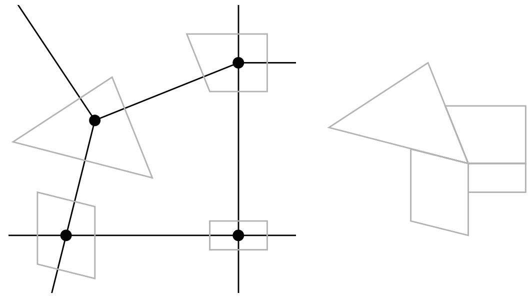


Figure 4. Vertices surrounding a face, with dual polygons. When the polygons are all in standard orientation, as shown here, they can slide together to fit around a single point, as shown on the right.

To make the whole tiling fit together, though, the tiles must fit at *every* tiling vertex, not just a single one. Note that the discussion of dualizing above leaves out the specification of edge lengths – a given network vertex could dualize to any number of different shapes, all with the same corner angles. It remains to find a way of choosing the edge lengths of the tiles so that all adjacent tiles fit together edge-to-edge.

Restricting vertices to make tiles fit together

The simplest solution is to force all tile edges to be of unit length. If we have a collection of polygonal tiles, all with unit length edges, and the angles round each tiling vertex total 360 degrees, then the tiles will fit together to cover the plane. The problem now is that unit edge-length tiles can not be created for all vertices – for some sets of angles, the polygon will fail to close and be a tile. This method is still useful, though, as the class of vertices that are dualizable is large. We now give some results concerning these vertices.

It is easy to see that all *regular* vertices (those where the angles between successive outgoing edges are all equal, like the spokes of a wheel) have unit-edge duals, namely, the regular polygons. (See Figure 5.) Therefore, we could create a complete tiling if we had a network where all the vertices were regular. Fortunately, there are such networks. They can be derived from a class of tilings called the *Laves tilings* [1], shown in Figure 6. Laves tilings are those where all vertices of the tiling are regular. There are fundamentally eleven different types (though one occurs in two distinct mirror-image forms). By placing network vertices at each vertex of a Laves tiling and connecting them with the appropriate edges, we can create a network whose unit-edge dual is a tiling.

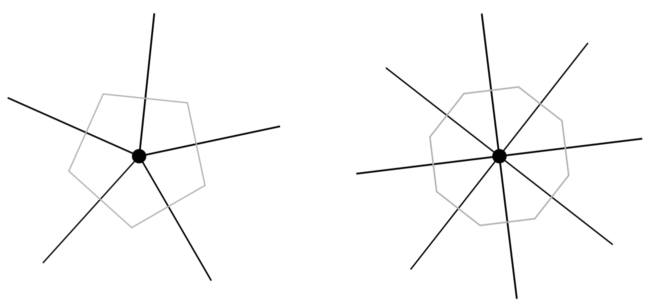


Figure 5. Two regular vertices, with corresponding unit-edge duals (which are regular polygons) shown.

Dualizing the Laves networks created this way gives the *Archimedean tilings* – the eleven possible periodic tilings composed entirely of regular polygons (see Figure 7). This is not yet a very interesting result – we have taken a known class of tilings, the Laves tilings, and used them to create networks which dualize to another known class of tilings, the Archimedean tilings. The real power of this method will come in the next section.

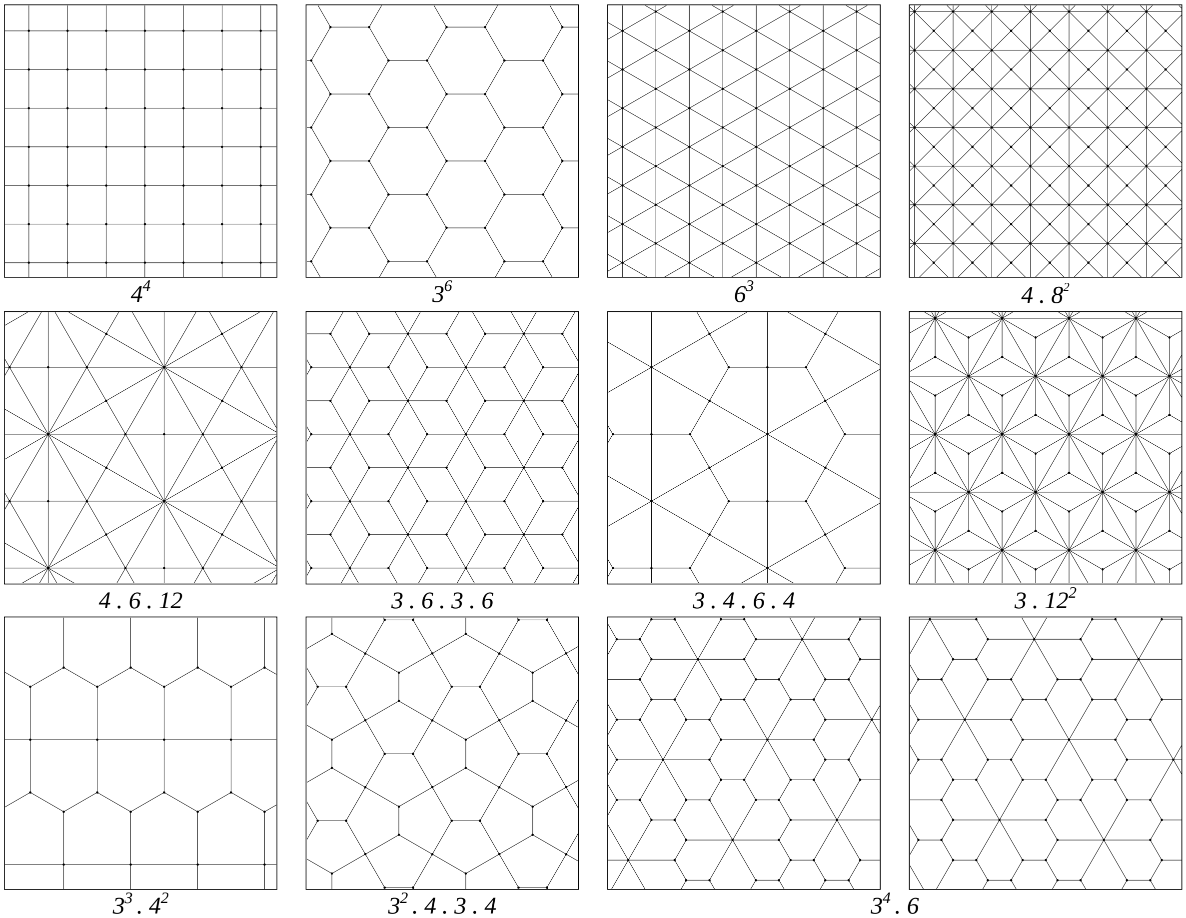


Figure 6. The eleven Laves networks (the last one occurs in two distinct mirror-image forms). All the vertices in these networks are regular, and therefore have unit-edge duals.

Creating new dualizable networks

We first define a mathematical property of the set of edges at a vertex that is necessary and sufficient for the unit-edge dual polygon to exist.

Theorem 1 *Let V be a network vertex with edges radiating from it at angles $\beta_1, \beta_2, \ldots, \beta_n$, where $\beta_1 \le \beta_2 \le \ldots \le \beta_n \le 2\pi$. V has a unit-edge dual tile if and only if the sum of the sines of angles $\beta_1, \ldots, \beta_n$ and the sum of the cosines of angles $\beta_1, \ldots, \beta_n$ are both zero.*

To see this, imagine drawing the dual tile with turtle graphics. Begin the turtle at the origin, with heading β_1, and move it one unit forward. Its position is now ($\cos \beta_1$, $\sin \beta_1$). Now rotate it left through an angle of $\beta_2 - \beta_1$ degrees, so it has a heading of β_2, and move it forward one unit again, to the point ($\cos \beta_1 + \cos \beta_2$, $\sin \beta_1 + \sin \beta_2$). After n steps, the turtle will be at the point with x equal to the sum of the cosines of the angles, and y equal to the sum of the sines. If this point is the origin (0,0), then the turtle will have traced out a closed polygon whose exterior angles are the differences between the successive β_is. Two examples of this procedure are shown in Figure 8. If the shape is closed, it is exactly the unit-edge dual of the polygon.

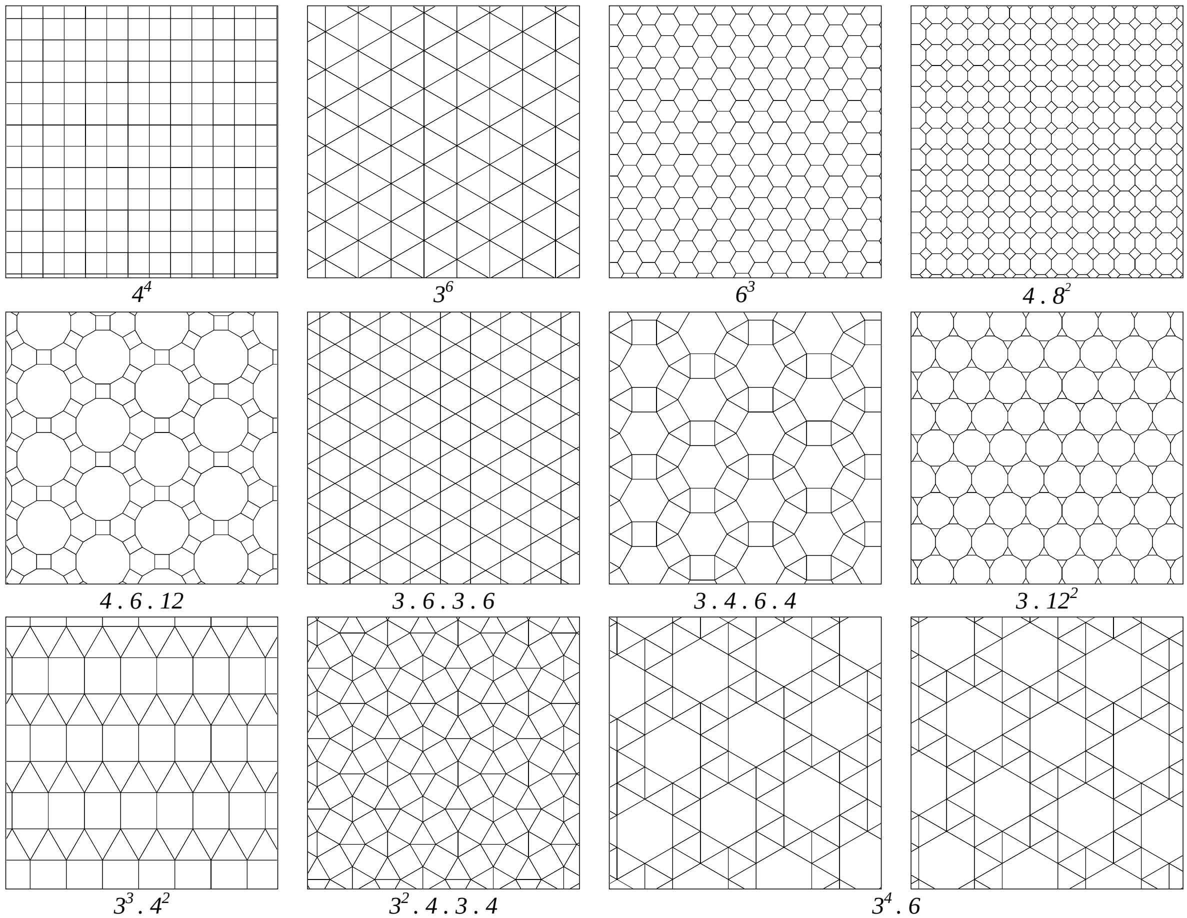


Figure 7. The eleven Archimedean tilings (the last one has two distinct mirror-image forms). Each tiling is the dual of the corresponding network of Figure 6.

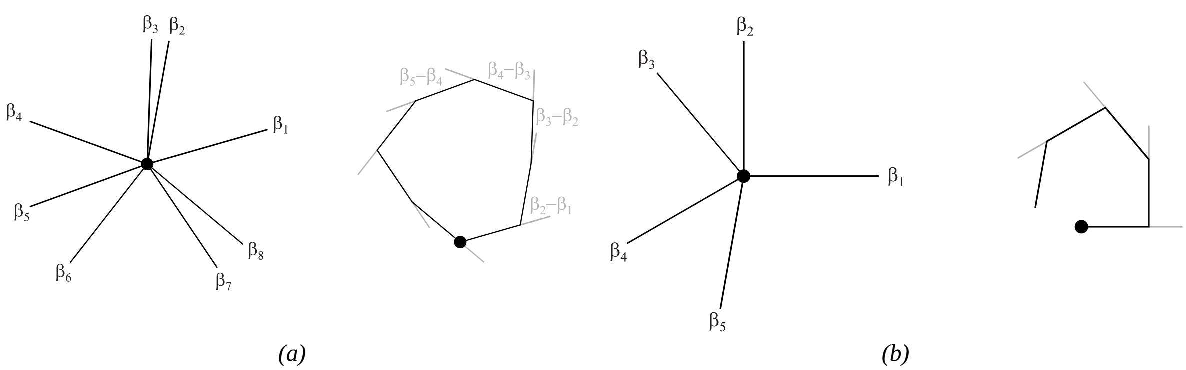


Figure 8. Two vertices, with the path traced out by the dualizing turtle. The turtle takes a unit step in direction $\beta 1$, then a step in direction $\beta 2$, and so on. After k steps, the turtle will be at ($\cos \beta_1 + \cos \beta_2 + \ldots + \cos \beta_k$, $\sin \beta_1 + \sin \beta_2 + \ldots + \sin \beta_k$). If, after n steps, the turtle is back at the origin (as in part (a)), then the figure closes and we have created a polygon with unit edges and exterior angles equal to the angles between successive vertex edges. If the figure doesn't close, as in part (b), then the vertex does not have a unit-edge dual.

Since we already know regular vertices are dualizable, we can use this result to show that certain types of nonregular vertices are also dualizable.

Theorem 2 *If two unit-edge dualizable networks are overlaid to create a new network, the resulting network is also unit-edge dualizable.*

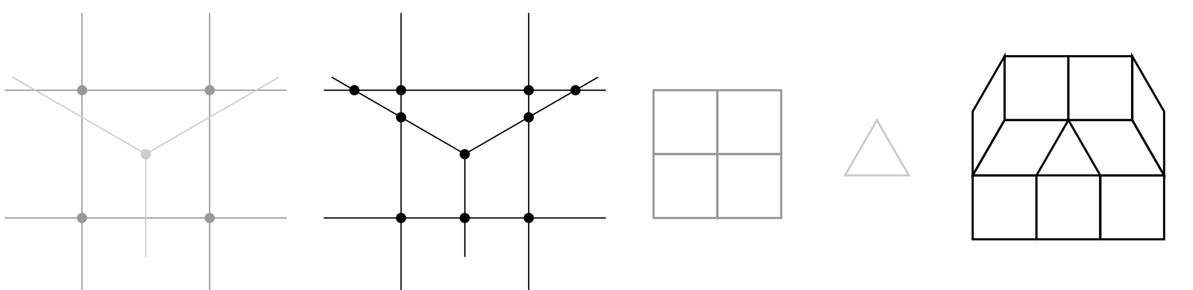
To overlay two networks, we union the two sets of vertices and edges together, inserting new vertices into edges as necessary to prevent edges from crossing. We also merge coincident vertices together. Note that edges do not get merged – it is possible to wind up with two or more edges leaving a vertex at the same angle. To show that the overlay process preserves unit-edge dualizability, we examine the three possible types of "interaction" between the two source networks.

1. **Coincident vertices.** If a vertex of one network falls exactly on a vertex of the other, we merge them into a single vertex with the union of the outgoing edge sets. If both source vertices are dualizable, then each has a zero sum of cosines of edge set angles (by the first theorem). The sum of the cosines of the union edge set angles is simply the two source sums added together. Since these two sums are both zero, their total is zero. The same argument applies in calculating the sum of the sines. By the first theorem again, then, the merged vertex with the combined outgoing edge set is dualizable.

2. **Edge on vertex.** If a vertex of one network lies on an edge of the other, we insert a degree-2 vertex into the edge at the point of intersection. A vertex inserted into an edge will be regular since it has two outgoing edges, 180 degrees apart. We now have two coincident vertices, both dualizable, and proceed as in the first case.

3. **Edge crossing edge.** If edges of the two networks intersect, we insert a vertex into each network at the intersection point. These coincident inserted vertices are both regular and therefore dualizable, so we can then merge them as before.

This overlay process is shown in Figure 9(a). Overlaying the light-gray network (the single vertex with three edges) onto the dark gray network requires adding vertices where edges of the two networks cross. The above results guarantee that all the vertices in the resulting network have unit-edge dual polygons. Since these polygons all have a common edge length, and the polygons corresponding to vertices surrounding any single face of the network will fit together at a point, the whole set of polygons will fit together to tile a patch of the plane.



(a) *(b)*

Figure 9. Inserting vertices to overlay two networks (part (a)), and showing the effect on the dual tilings (part (b)). All the tiles of the original two duals are preserved, but they are spread apart and interlaced, with rhombs (corresponding to the newly inserted vertices) filling in the gaps. Tiles of the original duals will not appear in the overlay tiling only when the corresponding vertex falls exactly on an edge or vertex of the other network.

Overlays produce interesting networks

We can now take two or more Laves networks, apply various transforms that preserve the regularity of vertices (rotation, translation, and uniform scaling), then overlay them and dualize the result to create novel tilings. The tilings created will be, in many cases, the Archimedean tilings associated with each source Laves network, interlaced, with rhombs filling in the gaps. In Figure 10, we show a network created by this overlay method. (It is actually a small portion of the network whose dual is Figure 11(c) – the degree-12 vertex in the lower left potion of the network corresponds to the large irregular dodecagon at the center of the tiling.)

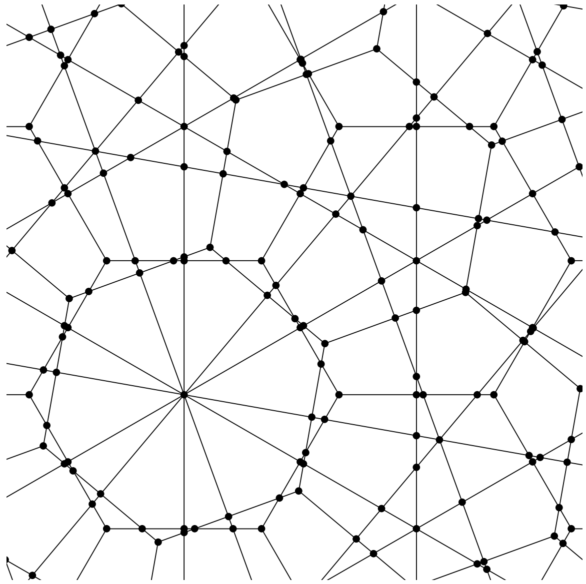


Figure 10. A portion of the network corresponding to the tiling in Figure 11(c). Since it is created by overlaying Laves networks, we know that all the vertices of this network have unit-edge dual polygons, which will fit together into a tiling.

Figure 11(a) shows a tiling created by overlaying the two mirror images of the $3^4.6$ Laves networks onto each other. To produce Figure 11(b), we overlaid four copies of the simple 4^4 Laves network atop one another, each rotated and scaled differently. The gray squares in the tiling correspond to vertices present in the initial networks, and the blue and white tiles are all the different sorts of rhomb used to pack the variously oriented gray squares together. Figure 11(c) shows a tiling produced by overlaying the 3.4.6.4 tiling onto a slightly rotated and scaled copy of itself. The irregular dodecagon at the center is produced by two degree-6 vertices in the original networks coinciding. Figure 11(d) shows the results of using a further generalization, not discussed in this paper, which allows the dual polygons to have non-unit edge lengths. The blue pentagons in this figure are actually the pentagons of the $3^2.4.3.4$ Laves tiling (see Figure 6), interlaced together with the regular polygons of the 4.6.12 Archimedean tiling (see Figure 7).

Conclusions and future work

We have developed a method of creating tilings through topological dualization of certain kinds of networks. We have further extended this method by using networks with weighted edges to create tilings with non-uniform edge lengths (as in Figure 11(d)), but space does not permit us to explore this generalization fully. There are also interesting possibilities for using this method to create tilings with nonconvex tiles.

Another interesting extension would be to develop animations showing how the tilings change as their underlying source networks are manipulated. Such animations would contain both continuously changing shapes and discrete retiling of local neighborhoods.

References

[1] Branko Grünbaum and G.C. Shephard. *Tilings and Patterns.* W.H. Freeman and Company, 1986.

[2] Peter Stampfli. New quasiperiodic lattices from the grid method. In István Hargittai, editor, *Quasicrystals, Networks, and Molecules of Fivefold Symmetry*, chapter 12, pages 201–221. VCH Publishers, 1990.

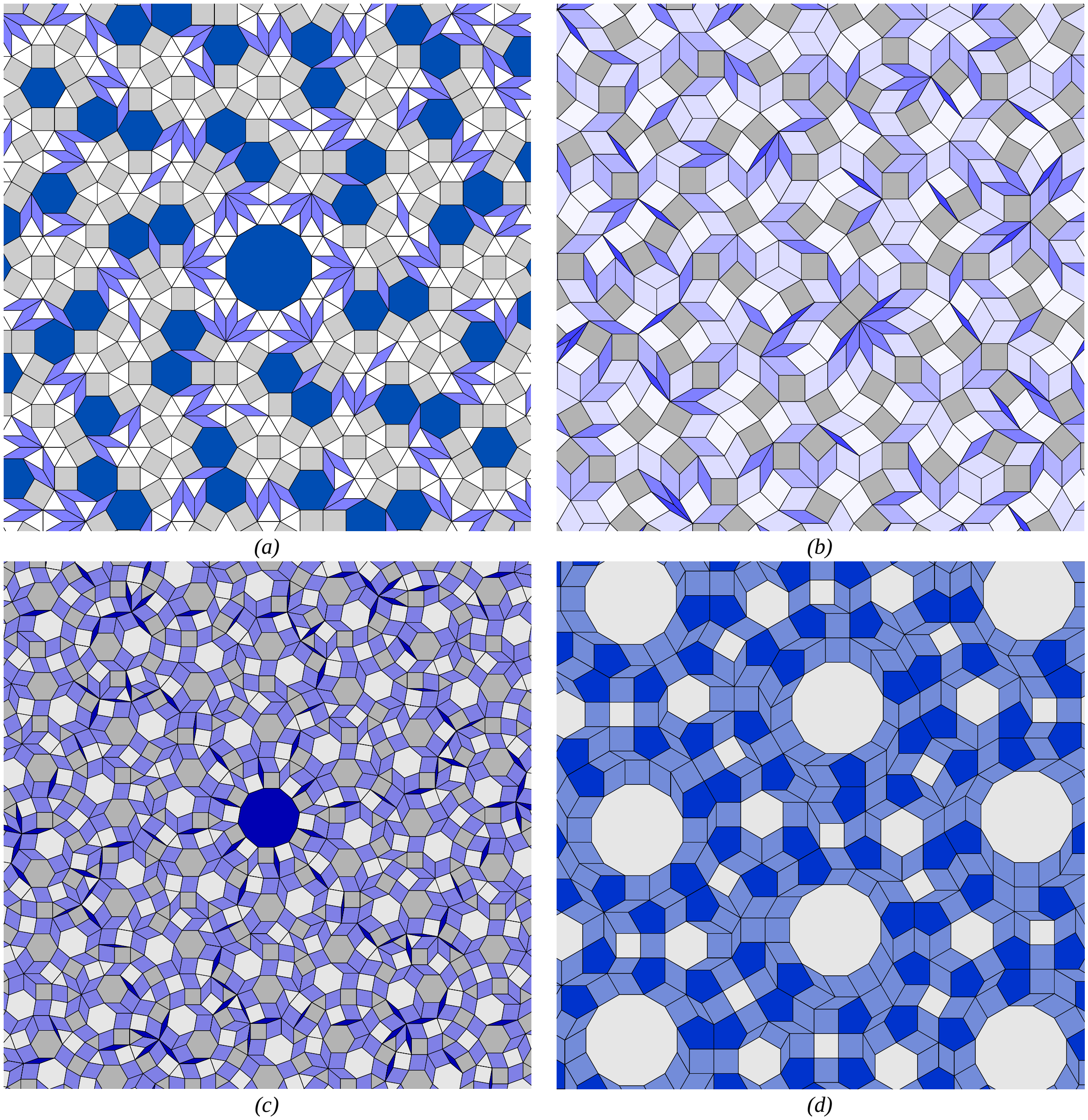


Figure 11. Four tilings created with the overlay/dual method.